An unconstrained optimization approach for finding real eigenvalues of even order symmetric tensors

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Abstract

Let n be a positive integer and m be a positive even integer. Let $\mathcal{A}$ be an mth order n-dimensional real weakly symmetric tensor and $\mathcal{B}$ be a real weakly symmetric positive definite tensor of the same size. $\lambda \in \mathbb{R}$ is called a $\mathcal{B}_r$-eigenvalue of $\mathcal{A}$ if $\mathcal{A}x^{m-1} = \lambda \mathcal{B}x^{m-1}$ for some $x \in \mathbb{R}^n \backslash \{0\}$. In this paper, we introduce two unconstrained optimization problems and obtain some variational characterizations for the minimum and maximum $\mathcal{B}_r$–eigenvalues of $\mathcal{A}$. These unconstrained optimization problems can be solved using some powerful optimization algorithms, such as the BFGS method. We provide some numerical results to illustrate the effectiveness of this approach for finding a Z-eigenvalue and for determining the positive semidefiniteness of an even order symmetric tensor.

Key words. Weakly symmetric tensors, tensor eigenvalues, positive semidefiniteness, unconstrained optimization.

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1 Introduction

Since the pioneering works of Qi [17] and Lim [13], the tensor eigenproblem has become an important part of numerical multilinear algebra. In this paper, we consider the real eigenvalue problems for even order real symmetric tensors. Eigenvalues of symmetric tensors have found applications in

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several areas, including automatic control, statistical data analysis, higher order diffusion tensor imaging, and image authenticity verification, etc., see for example, [17, 18, 20, 21, 22].

Throughout this paper, we assume that $\mathbb{R}$ is the real field, n is a positive integer, and m is a positive even integer. An mth-order n-dimensional real tensor

$$\mathcal{A} = (A_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{n \times n \times \cdots \times n}$$

is called symmetric if its entries are invariant under any permutations of their indices [10, 17]. A tensor $\mathcal{A}$ is called positive definite (positive semidefinite) if the multilinear form

$$\mathcal{A}x^m = \sum_{i_1, \cdots, i_m = 1}^{n} A_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}$$

is positive (nonnegative) for all $x \in \mathbb{R}^n \backslash \{0\}$. The notation $\mathcal{A}x^{m-1}$ denotes the vector in $\mathbb{R}^n$ whose i^{th} entry is

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, \cdots, i_m = 1}^{n} A_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}.$$

Following [6], $\mathcal{A}$ is called weakly symmetric if the gradient of $\mathcal{A}x^m$

$$\nabla (\mathcal{A}x^m) = m \mathcal{A}x^{m-1}$$

for all $x \in \mathbb{R}^n$. If $\mathcal{A}$ is symmetric, then it is weakly symmetric [6].

Various definitions of real eigenpairs for tensors have been introduced in the literature, including H-eigenvalues [17], Z-eigenvalues [17], and D-eigenvalues [20]. We use the following generalized eigenvalue definition, which includes the H-, Z-, and D-eigenvalues as special cases.

DEFINITION 1 *Let $\mathcal{A}$ and $\mathcal{B}$ be m^{th}-order n-dimensional real weakly symmetric tensors. Assume further that $\mathcal{B}$ is positive definite. If there exist a scalar $\lambda \in \mathbb{R}$ and a nonzero vector $x \in \mathbb{R}^n$ such that*

$$\mathcal{A}x^{m-1} = \lambda \mathcal{B}x^{m-1}, \tag{1.1}$$

then λ is called a $\mathcal{B}_r$–eigenvalue of $\mathcal{A}$ and x a $\mathcal{B}_r$–eigenvector with respect to λ. We denote the $\mathcal{B}_r$–spectrum of $\mathcal{A}$ by

$$\sigma_{\mathcal{B}_r}(\mathcal{A}) = \{\lambda : \lambda \text{ is a } \mathcal{B}_r-\text{eigenvalue of } \mathcal{A}\}.$$

REMARK 1 This definition was first introduced by Chang, Pearson, and Zhang [6] in a somewhat more general setting.

- If $\mathcal{B} = \mathcal{I} = (\delta_{i_1 i_2 \cdots i_m})$, the unit tensor, then $\mathcal{B}$ is weakly symmetric positive definite. Moreover, $\mathcal{B}x^m = \|x\|_m^m = x_1^m + x_2^m + \cdots + x_n^m$, $\mathcal{B}x^{m-1} = m[x_1^{m-1}, x_2^{m-1}, \cdots, x_n^{m-1}]^T$, and the $\mathcal{B}_r$–eigenvalues are H-eigenvalues.

- If $\mathcal{B} = I_n^{m/2}$, the tensor product of $m/2$ copies of the unit matrix $I_n \in \mathbb{R}^{n \times n}$, then $\mathcal{B}$ is weakly symmetric positive definite. Moreover, $\mathcal{B}x^m = (x^T x)^{m/2} = (x_1^2 + x_2^2 + \cdots + x_n^2)^{m/2}$, $\mathcal{B}x^{m-1} = m(x^T x)^{\frac{m-2}{2}} x$, and the $\mathcal{B}_r$–eigenvalues are Z-eigenvalues.

- If $\mathcal{B} = D^{m/2}$, the tensor product of $m/2$ copies of the symmetric positive definite matrix $D \in \mathbb{R}^{n \times n}$, then $\mathcal{B}$ is weakly symmetric positive definite. Moreover, $\mathcal{B}x^m = (x^T D x)^{m/2}$, $\mathcal{B}x^{m-1} = m(x^T D x)^{\frac{m-2}{2}} D x$, and the $\mathcal{B}_r$–eigenvalues are D-eigenvalues.

Calculation of all eigenvalues of a high order ($m > 2$) tensor is difficult, unless m and n are small [17]. In certain circumstances, however, one only needs to compute the largest or smallest eigenvalue of a tensor. For instance, the smallest H-eigenvalue or Z-eigenvalue of an even order symmetric tensor $\mathcal{A}$ can be used to determine the positive definiteness/semidefiniteness of $\mathcal{A}$ [17]. For a nonnegative tensor, the Perron-Frobenius theory asserts that its largest H-eigenvalue is its spectral radius [5, 8].

Recently, Kolda and Mayo [11] have extended the high order power method for symmetric tensor eigenproblems of Kofidis and Regalia [10] by introducing a shift parameter α to compute Z-eigenvalues of symmetric tensors. With a suitable choice of α, the resulting method, SSHOPM, converges to a Z-eigenvalue of the tensor when applied to a symmetric tensor. The found Z-eigenvalue is not necessarily the largest or smallest Z-eigenvalue. The rate of convergence of the SSHOPM method is linear [11].

An alternative approach for computing the eigenvalues of a symmetric tensor is to solve the constrained optimization problem [13]

$$\min \mathcal{A}x^m \quad \text{s.t.} \quad \mathcal{B}x^m = 1, \tag{1.2}$$

or

$$\max \mathcal{A}x^m \quad \text{s.t.} \quad \mathcal{B}x^m = 1. \tag{1.3}$$

The Karush-Kuhn-Tucker points of Problem (1.2) or (1.3) give $\mathcal{B}_r$–eigenvalues and $\mathcal{B}_r$–eigenvectors of $\mathcal{A}$. If we are interested in obtaining one eigenvalue,

then these problems can be solved using a local constrained optimization solver [15]. Note that in each problem, the objective function and the constraint function are both polynomials. Therefore, a global polynomial optimization method can be used, if we are interested in finding the largest or smallest $\mathcal{B}_r$–eigenvalue.

A more attractive approach for computing eigenvalues of even order symmetric tensors, however, is to use unconstrained optimization. This is motivated by the works of Auchmuty [1, 2], in which he proposed some unconstrained variational principles for generalized symmetric matrix eigenvalue problems. In particular, he [2] considered the unconstrained optimization problems

$$\min_{x \in \mathbb{R}^n} g_1(x) = \frac{1}{4}(x^T B x)^2 + \frac{1}{2} x^T A x, \tag{1.4}$$

and

$$\min_{x \in \mathbb{R}^n} g_2(x) = \frac{1}{4}(x^T B x)^2 - \frac{1}{2} \mathcal{A} x^T x, \tag{1.5}$$

where $A \in \mathbb{R}^{n \times n}$ is a symmetric matrix and $B \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix. He proved that Problem (1.4) can be used to find the smallest generalized B-eigenvalue of A and Problem (1.5) can be used to find the largest generalized B–eigenvalue of A. In this paper, we will extend Auchmuty's unconstrained variational principles for symmetric matrix eigenproblems [2] to even order weakly symmetric tensors.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminary results that will be used to establish the main results in Section 3. In Section 3, we introduce two unconstrained optimization problems and obtain some variational characterizations for the minimum and maximum $\mathcal{B}_r$–eigenvalues of $\mathcal{A}$. In Section 4, we give some numerical results. Some final remarks are given in Section 5.

2 Preliminaries

We start with the existence of $\mathcal{B}_r$–eigenvalues of $\mathcal{A}$. The existence of H-eigenvalues and Z-eigenvalues of an even order symmetric tensor was first studied by Qi [17]. In [6], Chang, Pearson, and Zhang proved the existence of at least n $\mathcal{B}_r$–eigenvalues when $\mathcal{A}$ is weakly symmetric and $\mathcal{B}$ is weakly symmetric positive definite, which is summarized in the following

THEOREM 1 *([6]) Assume that $\mathcal{A}$ and $\mathcal{B}$ are m^{th}-order n-dimensional real weakly symmetric tensors and $\mathcal{B}$ is positive definite. Then $\mathcal{A}$ has at least n $\mathcal{B}_r$–eigenvalues, with n distinct pairs of $\mathcal{B}_r$–eigenvectors.*

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In [17], Qi proved the existence of the maximum and minimum H-eigenvalues and Z-eigenvalues. Using a similar argument, we can prove

THEOREM 2 *Assume that $\mathcal{A}$ and $\mathcal{B}$ are m^{th}-order n-dimensional real weakly symmetric tensors and $\mathcal{B}$ is positive definite. Then $\sigma_{\mathcal{B}_r}(\mathcal{A})$ is not empty. Furthermore, there exist $\lambda_{\min} \in \sigma_{\mathcal{B}_r}(\mathcal{A})$ and $\lambda_{\max} \in \sigma_{\mathcal{B}_r}(\mathcal{A})$ such that*

$$-\infty < \lambda_{\min} \leq \lambda \leq \lambda_{\max} < \infty, \quad \forall \, \lambda \in \sigma_{\mathcal{B}_r}(\mathcal{A}).$$

Proof: Since $\mathcal{B}$ is positive definite, the set $\{x \in \mathbb{R}^n : \mathcal{B}x^m = 1\}$ is compact [6]. We also notice that function $\mathcal{A}x^m$ is continuous. Thus, the constrained optimization problem (1.2) has a global minimizer $\underline{x}$ and the constrained optimization problem (1.3) has a global maximizer $\bar{x}$.

At the global minimizer $\underline{x}$ of problem (1.2), there is a scalar $\underline{\lambda}$ such that the KKT conditions

$$m\mathcal{A}\underline{x}^{m-1} = \underline{\lambda}m\mathcal{B}\underline{x}^{m-1} \tag{2.1}$$

hold. Clearly, $\underline{\lambda} \in \sigma_{\mathcal{B}_r}(\mathcal{A})$. The inner product of (2.1) with $\underline{x}$ gives

$$\mathcal{A}\underline{x}^m = \underline{\lambda}\mathcal{B}\underline{x}^m = \underline{\lambda}.$$

Since $\underline{x}$ is a global minimizer of problem (1.2),

$$\underline{\lambda} \leq \lambda, \quad \forall \lambda \in \sigma_{\mathcal{B}_r}(\mathcal{A}).$$

Therefore, we can set $\lambda_{\min} = \underline{\lambda}$. Similarly, we can establish the existence of $\lambda_{\max}$ by using the global maximizer $\bar{x}$. $\qquad\square$

We next consider a property of weakly symmetric positive definite tensors, which generalizes a similar property for symmetric positive definite matrices.

THEOREM 3 *Assume that $\mathcal{B}$ is an mth-order n-dimensional weakly symmetric positive definite tensor. Let $\mu > 0$ be the smallest H-eigenvalue of $\mathcal{B}$. Then*

$$\mathcal{B}x^m \geq \mu\|x\|_m^m, \quad \forall x \in \mathbb{R}^n, \tag{2.2}$$

where $\|x\|_m$ is the m-norm of x.

Proof: When $x = 0$, (2.2) obviously holds. According to Theorem 2, μ is the global minimum value of

$$\min \mathcal{B}x^m, \quad \text{s.t.} \quad \|x\|_m^m = 1.$$

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For any $x \in \mathbb{R}^n \backslash \{0\}$, we have

$$\mathcal{B}\left(\frac{x}{\|x\|_m}\right)^m \geq \mu.$$

This implies (2.2). □

Finally we recall that a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is *coercive* if

$$\lim_{\|x\| \to \infty} f(x) = +\infty.$$

A nice feature of coercive functions is summarized in the following

THEOREM 4 *([16]) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. If f is coercive, then f has at least one global minimizer. If, in addition, the first partial derivatives exist on $\mathbb{R}^n$, then f attains its global minimizers at its critical points.*

3 Unconstrained variational principles for the minimal and maximal $\mathcal{B}_r$ eigenvalues

We now generalize the unconstrained variational principles of Auchmuty [2] to even order weakly symmetric tensors. We first consider the unconstrained optimization problem

$$\min f_1(x) = \frac{1}{2m}(\mathcal{B}x^m)^2 + \frac{1}{m}\mathcal{A}x^m. \tag{3.1}$$

When $\mathcal{A}$ and $\mathcal{B}$ are weakly symmetric, the gradient of the objective function f_1 is

$$\nabla f_1(x) = (\mathcal{B}x^m)\mathcal{B}x^{m-1} + \mathcal{A}x^{m-1}. \tag{3.2}$$

The following theorem summarizes the properties of function f_1.

THEOREM 5 *Assume that $\mathcal{A}$ and $\mathcal{B}$ are m^{th}-order n-dimensional real weakly symmetric tensors and $\mathcal{B}$ is positive definite. Let $\lambda_{\min}$ be the smallest $\mathcal{B}_r$–eigenvalue of $\mathcal{A}$. Then*
(a) f_1 is coercive on $\mathbb{R}^n$.
(b) The critical points of f_1 are
    (i) $x = 0$; and
    (ii) any $\mathcal{B}_r$–eigenvector x of $\mathcal{A}$ associated with a $\mathcal{B}_r$–eigenvalue $\lambda < 0$ of

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$\mathcal{A}$ satisfying $\mathcal{B}x^m = -\lambda$.

(c) If $\lambda_{\min} < 0$, then f_1 attains its global minimal value

$$\min f_1(x) = -\frac{1}{2m}\lambda_{\min}^2$$

at any $\mathcal{B}_r$–eigenvector associated with the $\mathcal{B}_r$–eigenvalue $\lambda_{\min}$ satisfying $\mathcal{B}x^m = -\lambda_{\min}$.

(d) If $\lambda_{\min} \geq 0$, then $x = 0$ is the unique critical point of f_1 and the unique global minimizer of f_1 on $\mathbb{R}^n$.

Proof: (a) Since $\mathcal{B}$ is weakly symmetric positive definite, Theorem 3 asserts that

$$\mathcal{B}x^m \geq \mu\|x\|_m^m,$$

where $\mu > 0$ is the smallest H-eigenvalue of $\mathcal{B}$. This implies

$$f_1(x) \geq \frac{\mu^2}{2m}\|x\|_m^{2m} - \frac{1}{m}\mathcal{A}x^m \to \infty$$

as $\|x\| \to \infty$. Thus, f_1 is coercive on $\mathbb{R}^n$.

(b) At a critical point of f_1, its gradient $\nabla f_1(x) = 0$, that is,

$$\mathcal{A}x^{m-1} = -(\mathcal{B}x^m)\mathcal{B}x^{m-1}. \tag{3.3}$$

Clearly, $x = 0$ is a critical point of f_1 as $\nabla f_1(0) = 0$. Moreover, if $\lambda < 0$ is a $\mathcal{B}_r$–eigenvalue of $\mathcal{A}$, then

$$\mathcal{A}x^{m-1} = \lambda\mathcal{B}x^{m-1}.$$

If $x \in \mathbb{R}^n\backslash\{0\}$ is a $\mathcal{B}_r$–eigenvector associated with this λ and satisfies $\mathcal{B}x^m = -\lambda$, then it is a critical point of f_1.

(c) From Theorem 2, $\lambda \geq \lambda_{\min}$, $\forall\lambda \in \sigma_{\mathcal{B}_r}(\mathcal{A})$. At the critical point $x \in \mathbb{R}^n\backslash\{0\}$ that is a $\mathcal{B}_r$–eigenvector associated with a $\mathcal{B}_r$–eigenvalue $\lambda < 0$ and satisfies $\mathcal{B}x^m = -\lambda$, $\mathcal{A}x^m = -\lambda^2$. Moreover,

$$f_1(x) = \frac{1}{2m}\lambda^2 - \frac{1}{m}\lambda^2 = -\frac{1}{2m}\lambda^2 \geq -\frac{1}{2m}\lambda_{\min}^2,$$

since $0 > \lambda \geq \lambda_{\min}$. According to Theorem 4 and part (b), f_1 attains the global minimum value $-\frac{1}{2m}\lambda_{\min}^2$ at any $\mathcal{B}_r$–eigenvector associated with the $\mathcal{B}_r$–eigenvalue $\lambda_{\min}$ satisfying $\mathcal{B}x^m = -\lambda_{\min}$.

(d) $\lambda_{\min} \geq 0$ implies that $\lambda \geq 0$ for any $\lambda \in \sigma_{\mathcal{B}_r}(\mathcal{A})$. Thus, $\mathcal{B}x^m = -\lambda$ does not hold for any $\mathcal{B}_r$–eigenvector x of $\mathcal{A}$ associated with a $\mathcal{B}_r$–eigenvalue λ

of $\mathcal{A}$, as $\mathcal{B}x^m > 0$ for any $x \in \mathbb{R}^n \backslash \{0\}$ by the positive definiteness of $\mathcal{B}$. Hence, $x = 0$ is the unique critical point of f_1. It is also the unique global minimizer of f_1 according to Theorem 4. $\qquad\square$

We next consider the unconstrained optimization problem

$$\min f_2(x) = \frac{1}{2m}(\mathcal{B}x^m)^2 - \frac{1}{m}\mathcal{A}x^m. \qquad (3.4)$$

When $\mathcal{A}$ and $\mathcal{B}$ are weakly symmetric, the gradient of the objective function f_2 is

$$\nabla f_2(x) = (\mathcal{B}x^m)\mathcal{B}x^{m-1} - \mathcal{A}x^{m-1}. \qquad (3.5)$$

Using a similar argument in the proof of the properties of f_1, we can prove the following properties about f_2.

THEOREM 6 *Assume that $\mathcal{A}$ and $\mathcal{B}$ are m^{th}-order n-dimensional real weakly symmetric tensors and $\mathcal{B}$ is positive definite. Let $\lambda_{\max}$ be the largest $\mathcal{B}_r$-eigenvalue of $\mathcal{A}$. Then*
(a) f_2 is coercive on $\mathbb{R}^n$.
(b) The critical points of f_2 are at
(i) $x = 0$; and
(ii) any $\mathcal{B}_r$-eigenvector x of $\mathcal{A}$ associated with a $\mathcal{B}_r$-eigenvalue $\lambda > 0$ of $\mathcal{A}$ satisfying $\mathcal{B}x^m = \lambda$.
(c) If $\lambda_{\max} > 0$, then f_2 attains its global minimal value

$$\min f_2(x) = -\frac{1}{2m}\lambda_{\max}^2$$

at any $\mathcal{B}$-eigenvector associated with the $\mathcal{B}_r$-eigenvalue $\lambda_{\max}$ satisfying $\mathcal{B}x^m = \lambda_{\max}$.
(d) If $\lambda_{\max} \leq 0$, then $x = 0$ is the unique critical point of f_2. Moreover, it is the unique global minimizer of f_2 on $\mathbb{R}^n$.

Note that the functions f_1 and f_2 are polynomials of degree $2m$. A global polynomial optimization solver such as GloptiPoly3 [9] can be used to find the smallest $\mathcal{B}_r$-eigenvalue of $\mathcal{A}$ and the largest $\mathcal{B}_r$-eigenvalue of $\mathcal{A}$ by solving Problem (3.1) and Problem (3.4) respectively, provided that $\lambda_{\min} < 0$ and $\lambda_{\max} > 0$. If $\lambda_{\min} \geq 0$, solving Problem (3.1) does not result in the smallest $\lambda_{\min}$. In this case, we can solve the shifted problem

$$\min_{x \in \mathbb{R}^n} s_1(x, t) = \frac{1}{2m}(\mathcal{B}x^m)^2 + \frac{1}{m}(\mathcal{A} + t\mathcal{B})x^m, \qquad (3.6)$$

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for a suitable $t < 0$. If $t < -\lambda_{\min}$, then the global minimum value of Problem (3.6) is $-\dfrac{1}{2m}(\lambda_{\min} + t)^2$. Thus, $\lambda_{\min}$ can be obtained by finding the global minimum of Problem (3.6). When $\lambda_{\max} < 0$, we can similarly solve the shifted problem

$$\min_{x \in \mathbb{R}^n} s_2(x, t) = \frac{1}{2m}(\mathcal{B}x^m)^2 - \frac{1}{m}(\mathcal{A} + t\mathcal{B})x^m, \tag{3.7}$$

for a suitable $t > -\lambda_{\max} \geq 0$ to find $\lambda_{\max}$.

Problem (3.1) can be used to determine whether an even order symmetric tensor $\mathcal{A}$ is positive semidefinite or not. Take $\mathcal{B} = \mathcal{I}$ or $\mathcal{B} = I_n^{m/2}$. If the global minimum value of f_1 equals 0, then $\mathcal{A}$ is positive semidefinite (or definite); otherwise, it is not. Assume that we have been able to determine that $\mathcal{A}$ is positive semidefinite. To further determine whether $\mathcal{A}$ is positive definite or semidefinite, we can solve (3.6) using $t = -1$. If the global minimum of s_1 is $-\frac{1}{2m}$, then $\mathcal{A}$ is only positive semidefinite; otherwise $\mathcal{A}$ is positive definite.

Local unconstrained optimization methods can be used to solve Problems (3.1) and (3.4). These methods do not guarantee finding a global minimum. However, they converge to a critical point (see for example, [15]). According to Theorems 5 and 6, the found nonzero critical point corresponds to a $\mathcal{B}_r$–eigenvalue of $\mathcal{A}$. Therefore, local optimization solvers have the ability to find other eigenvalues besides the extreme ones. Moreover, if solving Problem (3.1) with $\mathcal{B} = I_n^{m/2}$ or $\mathcal{B} = \mathcal{I}$ results in a nonzero critical point, then it corresponds to a negative Z-eigenvalue or H-eigenvalue. This implies that local unconstrained optimization solvers can be used to solve Problem (3.1) or Problem (3.6) to determine if $\mathcal{A}$ is positive semidefinite. Finally, a local unconstrained optimization method such as the BFGS method has a fast rate of convergence - which is superlinear.

4 Numerical results

In this section, we present some numerical results to illustrate the effectiveness of using the unconstrained variational principles for finding real eigenvalues of even order symmetric tensors. The experiments were done on a Laptop with an i3 CPU and a 4GB RAM, using MATLAB7.8.0 [14], the MATLAB Optimization Toolbox [14], and the Tensor Toolbox [3]. We did two groups of experiments: First, comparing the new approach with the SSHOPM method and the constrained optimization approach. Second, test-

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ing the ability of the new approach to determine positive semidefiniteness of even order symmetric tensors.

4.1 Effectiveness of finding a Z-eigenvalue

In our first group of experiments, we tested the new approach on finding Z-eigenvalues $(\mathcal{B} = I_n^{m/2})$ in order to compare it with the SSHOPM method ([11]). We will focus on solving Problem (3.1). The numerical behavior of solving Problem (3.4) is similar. When $\mathcal{B} = I_n^{m/2}$, the unconstrained variational principle (3.1) becomes

$$\min f_1^Z(x) = \frac{1}{2m}(x^T x)^m + \frac{1}{m}\mathcal{A}x^m. \tag{4.1}$$

The gradient of the corresponding objective is

$$\nabla f_1^Z(x) = (x^T x)^{m-1}x + \mathcal{A}x^{m-1}. \tag{4.2}$$

We tested the symmetric 4th order tensors defined in the following examples:

EXAMPLE 1 The 4th order n-dimensional tensor $\mathcal{A}$ is defined by

$$\mathcal{A}(i,j,k,l) = \begin{cases} -0.9, & \text{if } i = j = k = l; \\ 0.1, & \text{otherwise.} \end{cases} \quad 1 \le i,j,k,l \le n \tag{4.3}$$

EXAMPLE 2 The 4th order n-dimensional symmetric tensor $\mathcal{A}$ is generated as follows: First randomly generate tensor $\mathcal{T} = \texttt{randn}(n,n,n,n)$, then use the `symmetrize` function in the Matlab Tensor Toolbox [3] to symmetrize $\mathcal{T}$ and obtain $\mathcal{A} = \texttt{symmetrize}(\mathcal{T})$.

EXAMPLE 3 The 4th order n-dimensional symmetric tensor $\mathcal{A}$ is generated as follows: First randomly generate tensor $\mathcal{Y} = \texttt{randn}(n,n,n,n)$; then create tensor $\mathcal{Z}$ by setting $\mathcal{Z}(i,j,k,l) = \frac{1}{\mathcal{Y}(i,j,k,l)}$; and finally use the `symmetrize` function in the Matlab Tensor Toolbox [3] to symmetrize $\mathcal{Z}$ and obtain $\mathcal{A} = \texttt{symmetrize}(\mathcal{Z})$.

Since sometimes we are interested in finding the extreme eigenvalues of a tensor, we used the global polynomial optimization solver `GloptiPoly3` of Henrion, Lasserre, and Löfberg [9] to solve Problem (4.1) for some 4th order symmetric tensors in Examples 1–3. We observed that `GloptiPoly3` was able to solve (4.1) when $n \le 7$. When $n \ge 8$, it was unable to solve

(4.1) due to its memory requirement exceeding the capacity of the laptop computer we used.

From now on in this subsection we shall focus on solving (4.1) using a local optimization method. Specifically, we used the local optimization solver `fminunc` from the Matlab Optimization Toolbox [14] to solve Problem (4.1), with its default settings except for the following:

$$\texttt{GradObj}:\texttt{on},\ \texttt{LargeScale}:\texttt{off},\ \texttt{TolX} = \texttt{TolFun} = 10^{-12},\ \texttt{MaxIter} = 1000.$$
(4.4)

We tested this approach and compared it with two other approaches on some tensors from Examples 1–3.

4.1.1   Comparison with the constrained variational principle (1.2)

We tested and compared the unconstrained variational principle (4.1) with the constrained variational approach (1.2) (using $\mathcal{B} = I_n^{m/2}$) for finding Z-eigenvalues of some 4th order n-dimensional tensors given in Examples 1 and 2. For the constrained variational principle approach, we used the `fmincon` function from the Matlab Optimization Toolbox [14] to solve Problem (1.2), with its default settings except for the following:

$$\texttt{GradObj}:\texttt{on},\ \texttt{GradConstr}:\texttt{on},\ \texttt{TolX} = \texttt{TolFun} = 10^{-12},\ \texttt{MaxIter} = 1000.$$
(4.5)

Our numerical experiments indicated that Problem (1.2) can be a surprisingly difficult problem for `fmincon` when $m = 4$. Take the tensor $\mathcal{A}$ in Example 1 with $n = 4$ as an example. We ran `fminunc` on this tensor, using randomly generated initial vectors $x_0 = \texttt{randn}(4, 1)$ 100 times and using normalized randomly generated initial vectors $x_0 = y_0/\|y_0\|_2$ 100 times, where $y_0 = \texttt{randn}(4, 1)$. We also ran `fmincon` on this tensor similarly. We observed that

- Solving (4.1) via `fminunc`: In all of 200 runs, this method successfully found the Z-eigenvalue $\lambda = -0.9345$ of $\mathcal{A}$.

- Solving (1.2) with $\mathcal{B} = I_n^{m/2}$ via `fmincon`: (1) In the 100 runs using randomly generated initial vectors, this method successfully found the Z-eigenvalue $\lambda = -0.9345$ of $\mathcal{A}$ 11 times and it failed to find a Z-eigenvalue of $\mathcal{A}$ in 89 runs. (2) In the 100 runs using normalized randomly generated initial vectors, it successfully found the Z-eigenvalue $\lambda = -0.9345$ of $\mathcal{A}$ 43 times and it failed to find a Z-eigenvalue of $\mathcal{A}$ in 57 runs. The failures in both case (1) and case (2) were due to the divergence of the method.

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4.1.2 Comparison with the SSHOPM method

We now compare the unconstrained variational principle (4.1) with the SSHOPM method of Kolda and Mayo [11] for finding Z-eigenvalues of some 4th order n-dimensional tensors. The SSHOPM method is implemented as the `sshopm` function in the Matlab Tensor Toolbox [3]. We used the default settings of `sshopm` except for the following:

$$\texttt{Tol} = 10^{-12}, \ \texttt{MaxIts} = 5000. \tag{4.6}$$

The iterates $x^{(k)}$ generated by `sshopm` keeps the norm $\|x^{(k)}\|_2 = 1$. When `fminunc` converges to a nonzero critical point $\tilde{x}$ of Problem (4.1) corresponding to a Z-eigenvalue $\lambda \leq 0$, $\|\tilde{x}\|_2^m = -\lambda$. To have a fair comparison of the two methods, we first normalized the nonzero vector $\tilde{x}$ obtained by `fminunc` at termination so that $\hat{x} = \tilde{x}/\|\tilde{x}\|_2$. We define the error term by

$$\hat{\epsilon} = \|\mathcal{A}\hat{x}^3 - \lambda\hat{x}\|_2, \tag{4.7}$$

where $\hat{x}$ is either the vector obtained by `sshopm` at termination or the normalized vector when solving (4.1) via `fminunc` at termination.

The SSHOPM method without shift (i.e., the original SHOPM method of Kofidis and Regalia [10]) can fail to find a Z-eigenvalue, see for example, [10, 11]. We observed this behavior in our numerical experiments. Kolda and Mayo [11] proved that if the shift parameter $\alpha < 0$ is negative enough (or $\alpha > 0$ is large enough), then $x^{(k)}$ generated by the SSHOPM method converges to a Z-eigenvector. However, the SSHOPM method slows down significantly when a very negative $\alpha < 0$ or very large $\alpha > 0$ is used [11]. Kolda and Mayo [11] found that using $\alpha = -2, -1, 1, 2$ worked well in their tests.

Since the unconstrained variational principle (4.1) leads to negative Z-eigenvalues of $\mathcal{A}$, we used the SSHOPM method with a negative shift parameter α. We solved (4.1) via `fminunc` and ran `sshopm` with $\alpha = -2$ on tensors of various dimensions from Example 1 and Example 3. For each tensor we tested, we ran each of `fminunc` and `sshopm` on the tensor 100 times, using a normalized randomly generated initial vector

$$x_0 = \frac{y_0}{\|y_0\|_2}, \tag{4.8}$$

where $y_0 = \texttt{randn}(n, 1)$ at each time. We report the numerical results in Tables 1 and 2, in which "CPU time" and "Accuracy" denote the "average CPU time (in seconds)" and "average $\hat{\epsilon} = \|\mathcal{A}\hat{x}^3 - \lambda\hat{x}\|_2$" respectively.

Table 1: Unconstrained variational principle vs SSHOPM on some tensors from Example 1 using normalized randomly generated initial vectors

Problem	Method	CPU time	Accuracy
$n = 10$	SSHOPM ($\alpha = -2$)	0.16	5.33×10^{-7}
	(4.1) via `fminunc`	0.12	2.12×10^{-9}
$n = 20$	SSHOPM ($\alpha = -2$)	0.24	4.64×10^{-7}
	(4.1) via `fminunc`	0.18	2.65×10^{-9}
$n = 30$	SSHOPM ($\alpha = -2$)	0.62	3.73×10^{-7}
	(4.1) via `fminunc`	0.34	3.70×10^{-9}
$n = 40$	SSHOPM ($\alpha = -2$)	1.69	4.05×10^{-7}
	(4.1) via `fminunc`	0.60	3.10×10^{-9}
$n = 50$	SSHOPM ($\alpha = -2$)	4.05	3.63×10^{-7}
	(4.1) via `fminunc`	1.36	3.30×10^{-9}
$n = 60$	SSHOPM ($\alpha = -2$)	8.24	3.08×10^{-7}
	(4.1) via `fminunc`	2.69	4.37×10^{-9}

Table 2: Unconstrained variational principle vs SSHOPM on some random tensors generated from Example 2 using normalized randomly generated initial vectors

Problem	Method	CPU time	Accuracy
$n = 10$	SSHOPM ($\alpha = -2$)	0.38	1.03×10^{-6}
	(4.1) via `fminunc`	0.18	9.98×10^{-9}
$n = 20$	SSHOPM ($\alpha = -2$)	0.89	1.33×10^{-6}
	(4.1) via `fminunc`	0.27	2.19×10^{-8}
$n = 30$	SSHOPM ($\alpha = -2$)	2.31	1.48×10^{-6}
	(4.1) via `fminunc`	0.53	3.65×10^{-8}
$n = 40$	SSHOPM ($\alpha = -2$)	5.87	1.61×10^{-6}
	(4.1) via `fminunc`	1.06	6.52×10^{-8}
$n = 50$	SSHOPM ($\alpha = -2$)	11.84	1.69×10^{-6}
	(4.1) via `fminunc`	2.21	8.88×10^{-8}
$n = 60$	SSHOPM ($\alpha = -2$)	24.28	1.79×10^{-6}
	(4.1) via `fminunc`	4.56	1.09×10^{-7}

Table 3: Unconstrained variational principle vs SSHOPM on a tensor generated from Example 3 with $n = 25$, using normalized randomly generated initial vectors

Method	Min/Max/Mean Accuracy	Min/Max/Mean CPU time
(4.1) via `fminunc`	$1.06 \times 10^{-5}/3.44 \times 10^{-2}/4.23 \times 10^{-3}$	0.36/1.14/0.63
SSHOPM ($\alpha = 0$)	$1.36 \times 10^4/4.02 \times 10^4/2.91 \times 10^4$	25.78/26.85/26.09
SSHOPM ($\alpha = -1$)	$1.08 \times 10^4/5.26 \times 10^4/3.73 \times 10^4$	25.96/28.58/26.78
SSHOPM ($\alpha = -2$)	$2.56 \times 10^4/5.09 \times 10^4/3.97 \times 10^4$	25.84/27.50/26.34
SSHOPM ($\alpha = -5$)	$7.30 \times 10^3/5.55 \times 10^4/3.58 \times 10^4$	25.79/27.11/26.25
SSHOPM ($\alpha = -10$)	$3.69 \times 10^3/4.55 \times 10^4/1.56 \times 10^4$	25.78/25.91/25.81
SSHOPM ($\alpha = -100$)	$1.53/4.12 \times 10^4/2.80 \times 10^4$	25.66/ 25.84 /25.75
SSHOPM ($\alpha = -1000$)	$6.50 \times 10^{-3}/1.09 \times 10^{-2}/8.79 \times 10^{-3}$	3.59/4.26/3.81

From Tables 1 and 2, we observe that although both the SSHOPM method and the unconstrained optimization principle (4.1) successfully find a Z-eigenvalue of $\mathcal{A}$ in all cases, solving (4.1) via `fminunc` on average uses less CPU time than `sshopm`, particularly when n is large. This is perhaps due to the superlinear convergence property of the BFGS algorithm used in `fminunc`.

A natural question is how to choose a suitable shift parameter α in `sshopm`. Using $\alpha = -2$ worked well for the problems considered in Tables 1 and 2. However, this choice is not a suitable one for some tensors from Example 3. We illustrate this in Table 3, in which we report the numerical results on a tensor from Example 3 with dimension $n = 25$. We ran `sshopm` with various shift parameters and `fminunc` on this tensor 10 times, using a normalized randomly generated initial vector x_0 as defined in (4.8) with $n = 25$ at each time. The notations are:

- Min/Max/Mean Accuracy denotes the minimum, maximum, and mean $\hat{\epsilon} = \|\mathcal{A}\hat{x}^3 - \lambda\hat{x}\|_2$.

- Min/Max/Mean CPU time denotes the minimum, maximum, and mean CPU time used.

Clearly for this example, $\alpha = 0, -1, -2, -5, -10, -100$ are not a suitable choice for the shift parameter. Although the tensor used in this test is artificial, the numerical results indicate that choosing a suitable shift parameter can be crucial for the success of the SSHOPM method.

We now summarize our comparison of the SSHOPM method and the unconstrained variational principle approach:

- The SSHOPM method can be used to find a complex Z-eigenvalue and can handle both even and odd order symmetric tensors [11].

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- The unconstrained variational principle approach (e.g. solving (4.1) via `fminunc`) can be faster than the SSHOPM method particularly when n is large.

- If the purpose is to find one Z-eigenvalue for a given tensor, choosing a suitable shift parameter can be crucial for the SSHOPM method. Although sometimes it needs to solve a shifted problem (3.6) or (3.7) (see the next subsection), choosing a suitable shift parameter is less critical for the unconstrained variational principle approach.

- A global polynomial optimization solver can be used to solve the optimization problems arisen in the unconstrained variational principles to find the largest or smallest eigenvalues. The found Z-eigenvalue by the SSHOPM method is not necessarily the largest or the smallest one.

4.2 Determining positive semidefiniteness

In some applications, it is important to determine if an even order symmetric tensor $\mathcal{A}$ is positive semidefinite (see, for example, [21, 22]). An attractive property of function (3.1) is that any of its nonzero critical point is a $\mathcal{B}_r$-eigenvector of $\mathcal{A}$ corresponding to a $\mathcal{B}_r$-eigenvalue $\lambda < 0$. This feature allows us to use a local optimization solver to determine the positive semidefiniteness of an even order symmetric tensor, since $\mathcal{A}$ is positive semidefinite if all of its H-eigenvalues and all of its Z-eigenvalues are nonnegative ([17]).

To test the effectiveness of using the unconstrained optimization approach to determine the positive semidefiniteness of even order symmetric tensors, we did some preliminary numerical experiments in which Problem (3.6) was solved via `fminunc`. The parameters for `fminunc` were the same as in (4.4). Note that (3.6) becomes (3.1) when $t = 0$. When $t \neq 0$, solving (3.6) leads to a $\mathcal{B}_r$-eigenvalue of the tensor $\mathcal{A} + t\mathcal{B}$. In this situation, subtracting t from the computed eigenvalue of $\mathcal{A} + t\mathcal{B}$ will result in a $\mathcal{B}_r$-eigenvalue of $\mathcal{A}$. We tested both $\mathcal{B} = \mathcal{I}$(corresponding to H-eigenvalues) and $\mathcal{B} = I_n^{m/2}$ (corresponding to Z-eigenvalues). We summarize the unconstrained optimization approach for determining the positive semidefiniteness of an even order symmetric tensor $\mathcal{A}$ in Algorithm 1.

ALGORITHM 1

Input: Tensor $\mathcal{A}$.
Step 0. Choose parameters $t < 0$, $\eta_2 > \eta_1 > 0$, and tensor $\mathcal{B} = \mathcal{I}$ or $\mathcal{B} = I_n^{m/2}$.
Step 1. Solve Problem (3.6) with parameter t approximately using a local optimization solver such as `fminunc`. Let $\tilde{x}$ and $\tilde{s}$ denote the approximate optimal solution and optimal objective value of (3.6) found by the optimization solver.
Step 2.

- If $\mathcal{B}\tilde{x}^m < \eta_1$, then $\mathcal{A}$ is positive definite, stop.

- If $\mathcal{B}\tilde{x}^m > \eta_2$, then set $\tilde{\lambda} = -\sqrt{-2m\tilde{s}} - t$ (which is a $\mathcal{B}_r$-eigenvalue of $\mathcal{A}$). If $\tilde{\lambda} \geq 0$, then $\mathcal{A}$ is positive semidefinite; otherwise $\mathcal{A}$ is not positive semidefinite, stop.

REMARK 2 If $\eta_1 \leq \mathcal{B}\tilde{x}^m \leq \eta_2$, then Algorithm 1 is inconclusive. In this case, we can use a different shift parameter $t < 0$ and repeat Step 1.

For comparison, we also tested the SSHOPM method. For this method, we used the same setting as in (4.6) except for that `MaxIts` was changed to `MaxIts` $= 10000$.

EXAMPLE 4 The 4th order n-dimensional symmetric tensor $\mathcal{A}$ is generated as follows: First randomly generate tensor $\mathcal{T} = \texttt{randn}(n, n, n, n)$, then use the `symmetrize` function in the Matlab Tensor Toolbox [3] to symmetrize $\mathcal{T}$ and obtain $\mathcal{Z} = \texttt{symmetrize}(\mathcal{T})$. Finally set

$$
\mathcal{A}(i, j, k, l) = \left\{ \begin{array}{ll} 1000, & \text{if } 1 \leq i = j = k = l \leq n - 1; \\ -1, & \text{if } i = j = k = l = n; \\ \mathcal{Z}(i, j, k, l), & \text{otherwise.} \end{array} \right. \tag{4.9}
$$

$\mathcal{A}$ is not positive semidefinite when $n \geq 2$.

EXAMPLE 5 The 4th order 3-dimensional tensor $\mathcal{A}$ is defined by $\mathcal{A}(1, 1, 1, 1) = 1$; $\mathcal{A}(2, 2, 2, 2) = 0$; $\mathcal{A}(3, 3, 3, 3) = -0.001$; and $\mathcal{A}(i, j, k, l) = 0$ for all other i, j, k, l. This tensor is not positive semidefinite.

We tested Algorithm 1 using $\mathcal{B} = \mathcal{I}$ or $\mathcal{B} = I_n^{m/2}$ and shift parameters $t = 0$ or $t = -1$ and compared them with the SSHOPM method with different shift parameters on some tensors from Examples 4 and 5. For each tensor, we ran each of these methods 100 times, using a normalized randomly generated initial vector as defined in (4.8) at each time. We used $\eta_1 = 10^{-10}$ and $\eta_2 = 10^{-4}$ in Algorithm 1.

We report the numerical results in Tables 4 and 5. In both tables, "Success rate" denotes the percentage of times where a negative eigenvalue was found (and therefore the corresponding method successfully determined that $\mathcal{A}$ is not positive semidefinite); "CPU time" denotes the average CPU time (in seconds); and "NIT" denotes the average number of iterations used by the SSHOPM method.

For the tensor generated from Example 4, the SSHOPM method always converged to a positive eigenvalue when $\alpha = -2$, $\alpha = -10$, $\alpha = -50$. When $\alpha = -100$, it converged to a negative eigenvalue in 34 out of 100 runs. It successfully found a negative eigenvalue when $\alpha = -500$ in all of the 100 runs, using an average CPU time of 30 seconds. On the other hand, Algorithm 1 using $\mathcal{B} = \mathcal{I}$ or $\mathcal{B} = I_n^{m/2}$ and $t = 0$ or $t = -1$ successfully found a negative eigenvalue in all runs, using much less CPU time.

For the tensor from Example 5, when $t = 0$, Algorithm 1 using $\mathcal{B} = \mathcal{I}$ correctly identified that $\mathcal{A}$ is not positive semidefinite 91% of times; Algorithm 1 using $\mathcal{B} = I_n^{m/2}$ was only successful 32% of times. The failures in both cases were due to that `fminunc` converged to the critical number $x = 0$ of (3.1). This is because `fminunc` is a local optimization solver. It only guarantees to converge to a critical point. We found that using a negative parameter t can significantly increase the success rate, particularly in the $\mathcal{B} = I_n^{m/2}$ case. Indeed, when $t = -1$ was used, Algorithm 1 successfully found a negative eigenvalue 98% of times in both $\mathcal{B} = \mathcal{I}$ and $\mathcal{B} = I_n^{m/2}$ cases. The two failure runs in each case were due to that the eigenvalue $\lambda = 0$ was found. In all 100 runs, the SSHOPM method (with $\alpha = -2$) terminated when the maximum allowed number of iterations (which is 10000) was reached. In 10 out of 100 runs, the method terminated at an approximate Z-eigenvalue very close to $\lambda = 0$ (and hence failed to correctly determine the positive semidefiniteness of $\mathcal{A}$). We plot the computed Z-eigenvalues by the SSHOPM method in the 100 runs in Figure 1. From this figure we observe that the SSHOPM method successfully determined that $\mathcal{A}$ is not positive semidefinite in less than 90% of times.

In summary, Algorithm 1 using $\mathcal{B} = \mathcal{I}$ or $\mathcal{B} = I_n^{m/2}$ and a negative shift parameter t is more efficient than the SSHOPM method on determining the

Table 4: Determining positive semidefiniteness using a tensor generated in Example 4, $n = 30$

Method	Shift parameter	Success rate	CPU time	NIT
Algorithm 1 ($\mathcal{B} = \mathcal{I}$)	$t = 0$	100	1.36	
Algorithm 1 ($\mathcal{B} = \mathcal{I}$)	$t = -1$	100	1.26	
Algorithm 1 ($\mathcal{B} = I_n^{m/2}$)	$t = 0$	100	0.99	
Algorithm 1 ($\mathcal{B} = I_n^{m/2}$)	$t = -1$	100	0.99	
SSHOPM	$\alpha = -2$	0	2.97	409.4
SSHOPM	$\alpha = -10$	0	2.34	310.1
SSHOPM	$\alpha = -50$	0	7.41	1111.2
SSHOPM	$\alpha = -100$	34	3.22	449.3
SSHOPM	$\alpha = -500$	100	30.00	4693.4

Table 5: Determining positive semidefiniteness using the tensor in Example 5

Method	Shift parameter	Success rate	CPU time	NIT
Algorithm 1 ($\mathcal{B} = \mathcal{I}$)	$t = 0$	91	0.28	
Algorithm 1 ($\mathcal{B} = \mathcal{I}$)	$t = -1$	98	0.62	
Algorithm 1 ($\mathcal{B} = I_n^{m/2}$)	$t = 0$	32	0.16	
Algorithm 1 ($\mathcal{B} = I_n^{m/2}$)	$t = -1$	98	0.47	
SSHOPM	$\alpha = -2$	≤ 90	34.58	10000

positive semidefiniteness of the tensors from Examples 4 and 5 we tested in terms of CPU time.

We now turn to the performance of Algorithm 1 on positive semidefinite (definite) tensors. We tested some tensors from Examples 6 and 7 using Algorithm 1 with $\mathcal{B} = \mathcal{I}$ and $t = -1$ and with $\mathcal{B} = I_n^{m/2}$ and $t = -1$. For each tensor we tested, we run each method 100 times, using a normalized randomly generated initial vector as defined in (4.8) at each time. We report the numerical results in Tables 6 and 7. In Table 6, "min $\mathcal{B}\tilde{x}^m$" denotes the smallest $\mathcal{B}\tilde{x}^m$ in 100 runs and "Success rate" denotes the percentage of times where the minimum eigenvalue $\lambda = 0$ was obtained. In Table 7, "min/max $\mathcal{B}\tilde{x}^m$" denotes the smallest and largest $\mathcal{B}\tilde{x}^m$ in 100 runs and "Success rate" denotes the percentage of times when the method correctly determined that $\mathcal{A}$ is positive definite. From Tables 6 and 7, we observe that Algorithm 1 using "$\mathcal{B} = \mathcal{I}$ and $t = -1$" or "$\mathcal{B} = I_n^{m/2}$ and $t = -1$" was able to efficiently determine the positive semidefiniteness of the tensors from Examples 6 and 7 we tested.

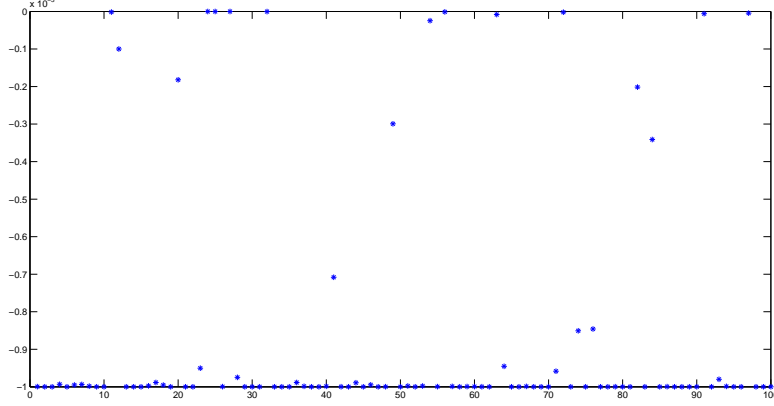**EXAMPLE 6** The 4th order n-dimensional tensor $\mathcal{A}$ is defined by $\mathcal{A}(k, k, k, k) =$

Figure 1: The computed Z-eigenvalues by the SSHOPM method in the 100 runs on tensor $\mathcal{A}$ from Example 5.

rand(1) for $k = 1, 2, \cdots, n-1$; $\mathcal{A}(n,n,n,n) = 0$; and $\mathcal{A}(i,j,k,l) = 0$ for all other i, j, k, l. $\mathcal{A}$ is positive semidefinite.

EXAMPLE 7 Consider the positive definite 4th order n-dimensional tensor $\mathcal{A}$ defined by $\mathcal{A}(k,k,k,k) = 10k$ for $k = 1, 2, \cdots, n$; and $\mathcal{A}(i,j,k,l) = 0$ for all other i, j, k, l.

Table 6: Determining positive semidefiniteness using a tensor in Example 6, $n = 30$

Method	min $\mathcal{B}\tilde{x}^m$	Success rate	CPU time
Algorithm 1 ($\mathcal{B} = \mathcal{I}$, $t = -1$)	1.00	100	1.68
Algorithm 1 ($\mathcal{B} = I_n^{m/2}$, $t = -1$)	1.00	100	2.54

Table 7: Determining positive semidefiniteness using a tensor in Example 7, $n = 30$

Method	min/max $\mathcal{B}\tilde{x}^m$	Success rate	CPU time
Algorithm 1 ($\mathcal{B} = \mathcal{I}$, $t = -1$)	7.12×10^{-17} / 2.65×10^{-15}	100	1.66
Algorithm 1 ($\mathcal{B} = I_n^{m/2}$, $t = -1$)	4.98×10^{-17} / 1.97×10^{-13}	100	1.81

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5 Final Remarks

We have introduced two unconstrained optimization problems and obtained some variational characterizations for the minimum and maximum $\mathcal{B}_r$ eigenvalues of an even order weakly symmetric tensor, where $\mathcal{B}$ is weakly symmetric positive definite. These unconstrained optimization problems can be solved using some powerful optimization algorithms, such as the BFGS method. We have provided some numerical results indicating that our approach of solving Problem (4.1) via `fminunc` compares favorably to the approach of solving (1.2) via `fmincon` and the SSHOPM method for finding a Z-eigenvalue of an even order symmetric tensor. This approach can also be used to find other $\mathcal{B}_r$ eigenvalues of even order symmetric tensors, including H-eigenvalues and D-eigenvalues and to find a $\mathcal{B}_r$ eigenvalue of an even order weakly symmetric tensor. Furthermore, we have provided some numerical results that show this approach is promising on determining positive semidefiniteness of an even order symmetric tensor.

A direction for future research is to develop a global polynomial optimization algorithm that can solve problems (3.1) and (3.4) with larger m and n.

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